

# Leading logarithmic large- $x$ resummation of off-diagonal splitting functions and coefficient functions

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## Abstract

We analyze the iterative structure of unfactorized partonic structure functions in the large- $x$  limit, and derive all-order expressions for the leading-logarithmic off-diagonal splitting functions  $P_{gq}$  and  $P_{qg}$  and the corresponding coefficient functions  $C_{\phi,q}$  and  $C_{2,g}$  in Higgs- and gauge-boson exchange deep-inelastic scattering. The splitting functions are given in terms of a new function not encountered in perturbative QCD so far, and vanish maximally in the supersymmetric limit  $C_A - C_F \rightarrow 0$ . The coefficient functions do not vanish in this limit, and are given by simple expressions in terms of the above new function and the well-known leading-logarithmic threshold exponential. Our results also apply to the evolution of parton fragmentation functions and semi-inclusive  $e^+e^-$  annihilation.

The splitting functions  $P_{ik}(x, \alpha_s)$ ,  $i, k = q, g$ , governing the scale dependence of the light-quark and gluon distributions of hadrons are among the most important quantities in perturbative QCD. In the helicity-averaged case these universal, but factorization-scheme dependent functions are completely known to the third order in the strong coupling constant  $\alpha_s$  [1–5]. Those results, as well as the computation of the second Mellin moment of the (flavour non-singlet) quark-quark splitting function to order  $\alpha_s^4$  [6], show a perturbative expansion which is remarkably well-behaved away from the momentum-fraction endpoints  $x = 0$  and  $x = 1$ . Without loss of information identifying the renormalization and mass-factorization scales, we write this expansion as

$$P_{ik}(x, \alpha_s) = \sum_{n=0}^{\infty} a_s^{n+1} P_{ik}^{(n)}(x) \quad \text{with} \quad a_s \equiv \frac{\alpha_s}{4\pi} . \quad (1)$$

In the small- $x$  (high-energy) limit all four flavour-singlet splitting functions exhibit a single-logarithmic higher-order enhancement, i.e., terms of the form  $\alpha_s^{n+1} x^{-1} \ln^{n-a} x$  occur at (almost) all orders with  $a \geq a_{\min} = 0$  for  $P_{gg}$  and  $P_{gq}$ , and  $a_{\min} = 1$  for  $P_{qq}$  and  $P_{qg}$  [7]. The contributions for  $a = 1$  have been obtained [8, 9] except for  $P_{qg}$ . Consequently only  $P_{gg}$  is known at next-to-leading logarithmic (NLL) small- $x$  accuracy at this point.

In this letter we address the large- $x$  (soft-gluon) limit. It is useful to switch to Mellin moments,

$$f(N) = \int_0^1 dx \, (x^{N-1} \{-1\}) f(x)_{\{+\}} , \quad (2)$$

where the parts in curly brackets refer to the case of  $(1-x)^{-1}$  +-distributions. Keeping only the leading – and subleading, if  $\ln^k N$  is replaced by  $\ln^k N + k\gamma_e \ln^{k-1} N$  at any stage – contributions, the relations between the relevant expressions in  $x$ -space and Mellin- $N$  space are given by

$$\frac{\ln^n(1-x)}{(1-x)_+} \stackrel{\text{M}}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \dots , \quad \ln^n(1-x) \stackrel{\text{M}}{=} \frac{(-1)^n}{N} \ln^n N + \dots \quad (3)$$

with  $\stackrel{\text{M}}{=}$  denoting equality under the Mellin transformation (2).

The dominant and subdominant ( $N^0$  and  $N^{-1}$ ) large- $x$  contributions to the diagonal splitting functions  $P_{qq}$  and  $P_{gg}$  in Eq. (1) are stable in the usual  $\overline{\text{MS}}$  factorization scheme [10], i.e., their form

$$P_{qq/gg}^{(n-1)}(N) = -A_{q/g}^{(n)} \ln N + B_{q/g}^{(n)} - C_{q/g}^{(n)} N^{-1} \ln N + \dots \quad (4)$$

is the same at all orders  $n$  [11, 12]. The quark and gluon cusp anomalous dimensions are related by  $A_g = C_A/C_F A_q$  [11], and the coefficients  $C^{(n)}$  are functions of lower-order quantities  $A^{(k)}$  [5, 12]. The  $1/N$ -suppressed off-diagonal splitting functions  $P_{gq}$  and  $P_{qg}$ , on the other hand, include a double-logarithmic higher-order enhancement with a particular colour structure,

$$\begin{aligned} C_F^{-1} P_{gq}^{(n)} = n_f^{-1} P_{qg}^{(n)} = & N^{-1} \ln^{2n} N D_0^{(n)} C_{AF}^n \\ & + N^{-1} \ln^{2n-1} N \left[ D_{1,AF}^{(n)} C_{AF} + D_{1,F}^{(n)} C_F + D_{1,T}^{(n)} n_f \right] C_{AF}^{n-1} + \dots . \end{aligned} \quad (5)$$

Here  $C_A$  and  $C_F$  are the usual  $\text{SU}(N_c)$  colour factors, which  $C_A = N_c = 3$  and  $C_F = 4/3$  in QCD.  $C_{AF} \equiv C_A - C_F$ , and  $n_f$  stands for the number of light flavours. All double logarithmic terms,  $\ln^k N$

with  $n+1 \leq k \leq 2n$  vanish for  $C_F = C_A$ , which is part of the colour-factor choice leading to an  $\mathcal{N} = 1$  supersymmetric theory. The leading coefficients  $D_0^{(n)}$  – which have the same modulus for  $P_{gq}$  and  $P_{qg}$  – vanish maximally, i.e., with the highest possible power of  $C_{AF}$ , the next-to-leading contributions  $D_1$  – which are not same for  $P_{gq}$  and  $P_{qg}$  – next-to-maximally etc. These properties and the coefficients  $D_j$  are known from the diagram calculations in Refs. [4, 5] to order  $\alpha_s^3$ .

Eq. (5) and the determination of the coefficients  $D_0$ ,  $D_1$  and  $D_2$  has been extended to order  $\alpha_s^4$  in Ref. [13]. Those results have been deduced from the – formally yet unproven – single-logarithmic large- $x$  behaviour of the physical evolution kernels  $K(N, \alpha_s)$  for the system  $(F_2, F_\phi)$  of flavour-singlet gauge-boson and Higgs exchange (in the heavy top-quark limit, see also Ref. [14]) structure functions in deep-inelastic scattering (DIS) [15],

$$\frac{dF}{d\ln Q^2} = \left( \beta(a_s) \frac{dC}{da_s} + CP \right) C^{-1} F \equiv KF \quad (6)$$

with

$$F = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2,q} & C_{2,g} \\ C_{\phi,q} & C_{\phi,g} \end{pmatrix}, \quad P = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\phi} \\ K_{\phi 2} & K_{\phi\phi} \end{pmatrix}, \quad (7)$$

in conjunction with the three-loop coefficient functions computed in Refs. [13, 16]. In particular it turned out that the fourth-order coefficient  $D_0^{(3)}$  vanishes, a fact that was attributed to an accidental cancellation of contributions. The single-log enhancement of the physical kernels provides relations between double-logarithmic contributions to the singlet splitting functions and coefficient functions also beyond this order but, unlike in corresponding non-singlet cases [17] which include Eq. (4) but not Eq. (5), no definite higher-order predictions of any expansion coefficients.

In the present letter we derive an all-order expression for the  $\alpha_s^{n+1} \ln^{2n}(1-x)$  leading-logarithmic (LL) large- $x$  contributions to the off-diagonal splitting functions  $P_{gq}$  and  $P_{qg}$ . This derivation is based on the large- $x$  properties of the unfactorized expressions for the respective gluon and quark contributions to the structure functions  $F_2$  and  $F_\phi$  in dimensional regularization. Hence we obtain the corresponding  $\alpha_s^n \ln^{2n-1}(1-x)$  contributions to the off-diagonal coefficient functions  $C_{2,g}$  and  $C_{\phi,q}$  in Eq. (7) as well. Our results also answer the questions whether or not  $D_0^{(3)} = 0$  in Eq. (5) is really accidental (it is not), and whether or not at least the leading double-logarithmic large- $x$  contributions to Eq. (6) definitely vanish at all orders in  $\alpha_s$  (they do).

The above-mentioned feature of the physical evolution kernel suggests an iterative structure of the unfactorized partonic structure functions or forward Compton amplitudes. For brevity suppressing, as already done in Eqs. (6) and (7) above, all functional dependences on  $N$ ,  $\alpha_s$  and the dimensional offset  $\epsilon$  with  $D = 4 - 2\epsilon$ , these quantities can be factorized as (cf., e.g., Refs. [13, 16])

$$T_{a,k} = \tilde{C}_{a,i} Z_{ik}. \quad (8)$$

Here the (process-dependent)  $D$ -dimensional coefficient functions  $\tilde{C}_{a,i}$  consists of contributions with all non-negative powers of  $\epsilon$ . The universal transition functions (or, in the language of the operator-product expansion (OPE), renormalization constants)  $Z_{ik}$  collecting all negative powers of  $\epsilon$  are related to the splitting functions in Eq. (7) (or the anomalous dimension  $\gamma$  of the OPE) by

$$-\gamma = P = \frac{dZ}{d\ln Q^2} Z^{-1}, \quad (9)$$

where we have, again without losing any information, identified the renormalization and factorization scale with the physical hard scale  $Q^2$ . Using the  $D$ -dimensional evolution of the coupling,

$$\frac{da_s}{d\ln Q^2} = -\varepsilon a_s + \beta(a_s) \quad (10)$$

where  $\beta(a_s)$  denotes the usual four-dimensional beta function of QCD,  $\beta(a_s) = -\beta_0 a_s^2 + \dots$  with  $\beta_0 = 11/3 C_A - 2/3 n_f$ , Eq. (9) can be solved for  $Z$  order by order in  $\alpha_s$ .

In general, the higher-order coefficients  $Z^{(n)}$  become very complicated at higher powers  $n$  of  $a_s$ . Here, however, we are interested only in the LL contributions at order  $N^{-1}$  for  $Z_{qg}$  and  $Z_{gq}$  and  $N^0$  for  $Z_{qq}$  and  $Z_{gg}$  (required for Eq. (8) also in the off-diagonal cases). Consequently there can be at most one off-diagonal  $N^{-1}$  factor per term, and  $P_{qq,gg}^{(n \geq 1)}$  can be neglected at leading- and also next-to-leading logarithmic (NLL) accuracy due to Eq. (4). Finally  $\beta(a_s)$  in Eq. (10) only enters at the NLL level. For the off-diagonal  $i \neq k$  entries these simplifications lead to

$$\begin{aligned} Z_{ik}^{(n)} \cong & \frac{1}{n!} \varepsilon^{-n} \sum_{l=0}^{n-1} \left( \gamma_{ii}^{(0)} \right)^{n-l-1} \gamma_{ik}^{(0)} \left( \gamma_{kk}^{(0)} \right)^l \\ & + \frac{1}{n!} \sum_{m=1}^{n-1} \varepsilon^{-n+m} \sum_{l=0}^{n-m-1} \frac{(m+l)!}{l!} \left( \gamma_{ii}^{(0)} \right)^{n-m-l-1} \gamma_{ik}^{(m)} \left( \gamma_{kk}^{(0)} \right)^l \end{aligned} \quad (11)$$

with, always keeping the LL contributions only,  $\gamma_{qq}^{(0)} = 4C_F \ln N$ ,  $\gamma_{gg}^{(0)} = 4C_A \ln N$ ,  $\gamma_{qg}^{(0)} = -2n_f N^{-1}$  and  $\gamma_{gq}^{(0)} = -2C_F N^{-1}$ . The corresponding diagonal quantities  $Z_{ii}^{(n)}$ ,  $i = q, g$ , are simply given by

$$Z_{ii}^{(n)} \cong \frac{1}{n!} \varepsilon^{-n} \left( \gamma_{ii}^{(0)} \right)^n. \quad (12)$$

Here and below  $\cong$  denotes equality if NLL contributions on both sides are neglected.

The  $\varepsilon^{-n} \dots \varepsilon^{-2}$  contributions at the  $n$ -th order in  $a_s$  of the products (8) include only lower-order quantities and thus provide a (at high  $n$  large) number of consistency checks. The  $\varepsilon^{-1}$  and  $\varepsilon^0$  terms include the desired  $n$ -loop contributions to the splitting functions and (four-dimensional) coefficient functions, respectively, in Eq. (7). In order to determine these quantities at a higher order  $a_s^l$ , also the coefficients of  $\varepsilon^k$  with  $0 < k \leq l - n$  are required. Hence an all-order determination of the splitting functions and coefficient functions requires expressions for  $T_{a,k}$  which are, at the logarithmic accuracy under consideration, exact in both  $\alpha_s$  and  $\varepsilon$ .

The first four  $\varepsilon^{-k}$  coefficients of the amplitudes  $T_{\phi,q}^{(n)}$  and  $T_{2,g}^{(n)}$  can be determined at all orders  $n$  from the third-order calculations in Refs. [5, 13, 16] and the all-order mass-factorization (or OPE) relation (8) with Eqs. (11) and (12). These results are of the form

$$\frac{1}{C_F} T_{\phi,q}^{(n)} \cong \frac{1}{n_f} T_{2,g}^{(n)} \cong \frac{\ln^{n-1} N}{N \varepsilon^n} \sum_{m=0}^{\infty} (\varepsilon \ln N)^m \mathcal{L}_{n,m} \left( C_F^{n-1} + C_F^{n-2} C_A + \dots + C_A^{n-1} \right), \quad (13)$$

i.e., the leading-logarithmic expansion coefficients  $\mathcal{L}_{n,m}$  at a given order in  $a_s$  and  $\varepsilon$  are the same for both off-diagonal amplitude and all contributing colour factors. Eq. (13) is the first of two equations with a clear-cut all- $\varepsilon$  structure to all orders in  $\alpha_s$  which, unavoidably, is guaranteed only to a

finite depth (here  $m = 3$ ) in  $\varepsilon$  by previous results. However, the simplicity of the structure and the tight functional forms of the  $D$ -dimensional expressions, see Eqs. (17) - (19) below, very strongly suggest that the all- $\varepsilon$  form is indeed correct. Also an inspection of the ladder-type diagrams generating the leading logarithmic large- $x$  contributions to  $T_{\phi,q}$ , illustrated in Fig. 1 (a), indicates that the  $\alpha_s^n C_F^{n-k} C_A^k$ ,  $0 < k < n$ , LL terms have the same coefficients as their  $\alpha_s^n C_F^n$  counterparts: any differences between different colour factors would be of a combinatorial nature, and thus be obvious from the known first powers in  $\varepsilon$ . The situation is analogous for the case of  $T_{2,g}^{(n)}$ , see Fig. 1 (b).

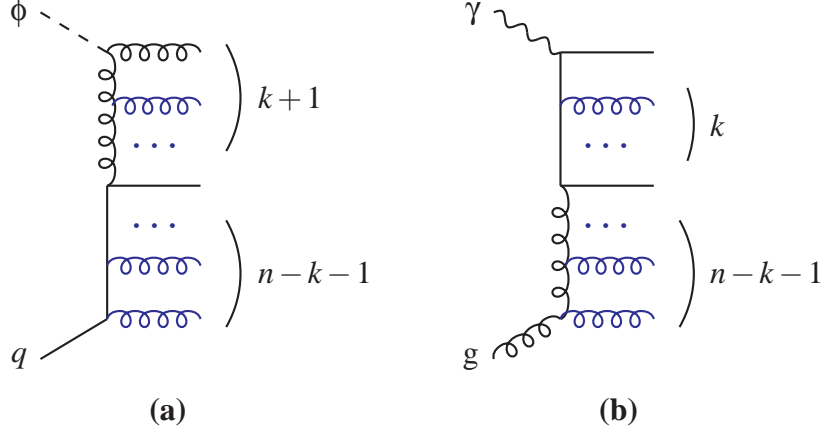


Figure 1: Typical diagrams for the leading-logarithmic large- $x$  terms of the  $n$ -th order quantities  $T_{\phi,q}^{(n)}$  (left) and  $T_{2,g}^{(n)}$  (right) in Eqs. (8) and (13). Shown are  $C_F^{n-k} C_A^k$  and  $n_f C_A^{n-k-1} C_F^k$  contributions to the former and latter expressions, respectively.

It is therefore sufficient to derive the complete LL expression for just one colour structure of one of the two quantities in Eq. (13) to all orders  $n$ . For this we choose the abelian  $C_F^n$  parts of  $T_{\phi,q}^{(n)}$ , as Fig. 1 (a) for  $k = 0$  suggests a factorization in terms of  $T_{\phi,q}^{(1)}$  and  $T_{2,q}^{(n-1)}$  for these quantities. Indeed, the third-order results of Ref. [13] imply

$$T_{\phi,q}^{(n)} \Big|_{C_F^n} \cong \frac{1}{n} T_{\phi,q}^{(1)} T_{2,q}^{(n-1)} \cong \frac{1}{n!} T_{\phi,q}^{(1)} \left( T_{2,q}^{(1)} \right)^{n-1}. \quad (14)$$

Here the second equality is due to

$$T_{2,q}^{(n)} \cong \frac{1}{n!} \left( T_{2,q}^{(1)} \right)^n \quad (15)$$

which, in conjunction with Eq. (19) below, is equivalent to the well-known leading-logarithmic threshold-exponentiation result [18]

$$C_{2,q} \cong \exp \left( 2a_s C_F \ln^2 N \right). \quad (16)$$

Eq. (14) is the second of the two all- $\varepsilon$  relations mentioned below Eq. (13), and the comments made there also apply here. Collecting the  $\alpha_s$ -expansion coefficients (14), one arrives at the closed all-order expression

$$T_{\phi,q} \Big|_{C_F \text{ only}} \cong T_{\phi,q}^{(1)} \frac{\exp \left( a_s T_{2,q}^{(1)} \right) - 1}{T_{2,q}^{(1)}} \quad (17)$$

in terms of the completely known  $D$ -dimensional one-loop quantities (see, e.g., Ref. [19]) with

$$T_{\phi,q}^{(1)} = -2C_F \frac{1}{\varepsilon} (1-x)^{-\varepsilon} \stackrel{M}{=} -\frac{2C_F}{N} \frac{1}{\varepsilon} \exp(\varepsilon \ln N), \quad (18)$$

$$T_{2,q}^{(1)} = -4C_F \frac{1}{\varepsilon} (1-x)_+^{-1-\varepsilon} + \text{virtual} \stackrel{M}{=} 4C_F \frac{1}{\varepsilon^2} (\exp(\varepsilon \ln N) - 1) \quad (19)$$

at leading-logarithmic accuracy in both  $x$ - and  $N$ -space. Together with Eq. (13) above, these three relations completely specify the LL contributions to  $T_{\phi,q}$  and  $T_{2,g}$  to all orders in  $\alpha_s$  and  $\varepsilon$ .

As the leading-log expression for  $T_{\phi,g}$  are completely analogous to Eqs. (15), (16) and (19) for  $T_{2,q}$ , we are now ready to perform the all-order mass factorization of  $T_{\phi,q}$  and  $T_{2,g}$ . We carry out this procedure via expanding all relevant expressions to a finite, but very high order using FORM [20], and finally deduce the all-order leading-logarithmic splitting functions and coefficient functions. Starting with  $T_{\phi,q}$ , the result for the former reads

$$P_{gq}^{\text{LL}}(N, \alpha_s) = \frac{C_F}{N} \frac{\alpha_s}{2\pi} \mathcal{B}_0(\tilde{a}_s), \quad \tilde{a}_s = \frac{\alpha_s}{\pi} (C_F - C_A) \ln^2 N \quad (20)$$

with

$$\mathcal{B}_0(x) = \sum_{n=0}^{\infty} \frac{B_n}{(n!)^2} x^n = 1 - \frac{x}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n!)^2} |B_{2n}| x^{2n}. \quad (21)$$

$B_n$  are the Bernoulli numbers in the standard normalization of Ref. [21]:  $B_{2n+1} = 0$  for  $n \geq 1$  and

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad \dots, \quad B_{12} = -\frac{691}{2730}, \quad \dots$$

The result for the corresponding coefficient function is given by

$$C_{\phi,q}^{\text{LL}}(N, \alpha_s) = \frac{1}{N} \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{2\pi} \right)^n \ln^{2n-1} N \sum_{a=1}^n C_F^a C_A^{n-a} \sum_{j=0}^n \frac{2^{j-1} B_j}{(j!)^2} \frac{(-1)^{j+a}}{(n-j)!} \binom{j-1}{a-1}. \quad (22)$$

The corresponding results for  $P_{qg}$  and  $C_{2,g}$  can be obtained from Eqs. (20) and (22) by replacing one power of  $C_F$  by  $n_f$ , and then interchange  $C_F$  and  $C_A$ . Consequently  $D_0^{(3)} = 0$  for both off-diagonal splitting functions in Eq. (5) is not at all accidental. In fact, the corresponding leading-log contributions vanish at all even orders in  $\alpha_s$ . Note also that Eq. (20) confirms the colour structure of Eq. (5) to all orders. We will provide a more transparent form of Eq. (22) below.

The function  $\mathcal{B}_0(x)$  in Eq. (21) appears to be new – at least it is not too widely known. Using the relation between the even- $n$  values of the Riemann  $\zeta$ -function and the corresponding Bernoulli numbers [21], it can be rewritten as

$$\mathcal{B}_0(x) = 1 - \frac{x}{2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \zeta_{2n} \left( \frac{x}{2\pi} \right)^{2n}. \quad (23)$$

The expansion coefficients in the sum differ from the Taylor coefficients of  $\cos(x/(2\pi))$  by the factor  $\zeta_{2n}$ . Hence, due to  $\zeta_{2n} \rightarrow 1$  for  $n \rightarrow \infty$ , the series (21) and (23) converge for all values on  $x$ . The sum entering  $\mathcal{B}_0(2\pi i)$  is known [22, 23], if not in a closed form [24].

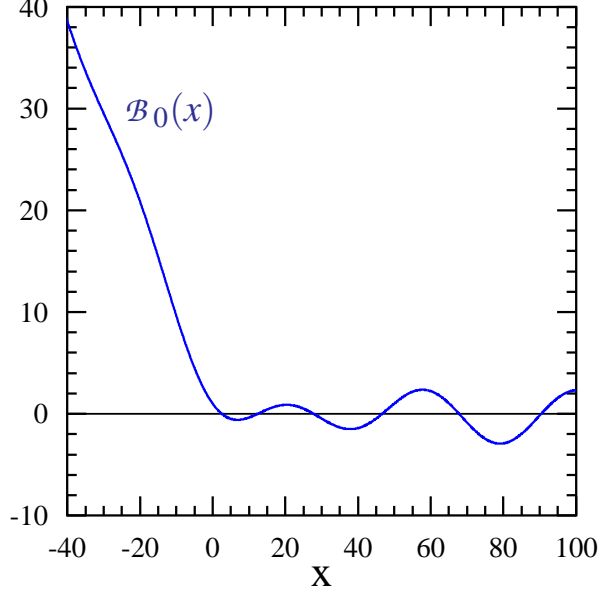


Figure 2: The function  $\mathcal{B}_0(x)$  in Eq. (21), evaluated using its defining Taylor expansion.

The numerical behaviour of  $\mathcal{B}_0(x)$  is illustrated in Fig. 2. The even part turns out to oscillate around  $|x|/2$ , resulting in oscillations around  $y = 0$  for positive and  $y = -x$  for negative values of  $x$ , respectively. Related functions, which we expect to enter a generalization of the present resummation to next-to-leading large- $N$  accuracy, are given by

$$\mathcal{B}_1(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+1)!} x^n, \quad \mathcal{B}_{-1}(x) = \sum_{n=1}^{\infty} \frac{B_n}{n!(n-1)!} x^n. \quad (24)$$

These functions are related to  $\mathcal{B}_0(x)$  by

$$\frac{d}{dx}(x\mathcal{B}_1) = \mathcal{B}_0, \quad \frac{d}{dx}\mathcal{B}_0 = \frac{1}{x}\mathcal{B}_{-1}. \quad (25)$$

Having determined the leading-log off-diagonal splitting functions and coefficient functions to all orders in  $\alpha_s$ , we are now in a position to prove (or disprove) the LL part of the conjecture [13] of the vanishing double-logarithmic contributions to the singlet physical kernel for the structure functions  $F_2$  and  $F_\phi$ . At this accuracy  $\beta(a_s)$  can be neglected in Eq. (6) above, leaving  $K \cong CPC^{-1}$  with

$$C^{-1} \cong \frac{1}{C_{2,q}C_{\phi,g}} \begin{pmatrix} C_{\phi,g} & -C_{2,g} \\ -C_{\phi,q} & C_{2,q} \end{pmatrix}, \quad P^{(n \geq 1)} \cong \begin{pmatrix} 0 & P_{qg}^{(n)} \\ P_{gq}^{(n)} & 0 \end{pmatrix}, \quad (26)$$

recall the notational convention below Eq. (12), which yields

$$K_{\phi 2} \cong (C_{2,q})^{-1} \left\{ C_{\phi,g} P_{gq} + C_{\phi,q} \alpha_s \left( P_{qq}^{(0)} - P_{gg}^{(0)} \right) \right\}, \quad (27)$$

$$K_{2\phi} \cong (C_{\phi,g})^{-1} \left\{ C_{2,q} P_{qg} + C_{2,g} \alpha_s \left( P_{gg}^{(0)} - P_{qq}^{(0)} \right) \right\}. \quad (28)$$

Inserting Eq. (16) and our results (20) and (22) into Eq. (27), and the corresponding relations into Eq. (28), the right-hand sides are indeed found to vanish at all orders  $\alpha_s^{n \geq 2}$ .



As all quantities entering Eq. (27) and Eq. (28) are known in closed forms (among which we now include  $\mathcal{B}_0$ ), these relations can now be used to cast Eq. (22) into the more transparent form

$$C_{\phi,q}^{\text{LL}}(N, \alpha_s) = \frac{1}{2N \ln N} \frac{C_F}{C_F - C_A} \left\{ \exp(2C_A a_s \ln^2 N) \mathcal{B}_0(\tilde{a}_s) - \exp(2C_F a_s \ln^2 N) \right\}, \quad (29)$$

where the two exponentials are the LL threshold expressions for  $C_{\phi,g}$  and  $C_{2,q}$  [18], and  $a_s$  and  $\tilde{a}_s$  have been defined in Eqs. (1) and (20). The corresponding result for  $C_{2,g}^{\text{LL}}$  is obtained from Eq. (29) by the colour-factor replacement given below Eq. (22) which includes  $\tilde{a}_s \rightarrow -\tilde{a}_s$ . Unlike the LL splitting functions, the coefficient functions do not vanish for  $C_F = C_A$  – but the curly bracket in Eq. (29) does, cancelling the corresponding pole in the prefactor.

To summarize, we have derived all-order expressions for the large- $x$ /large- $N$  leading logarithmic (LL) contributions to the off-diagonal splitting functions  $P_{gq}$  and  $P_{qg}$  and the corresponding coefficient functions  $C_{\phi,q}$  and  $C_{2,g}$  in Higgs- and gauge-boson exchange deep-inelastic scattering. Our results show that the LL coefficient for the former two quantities vanish at all even orders in the strong coupling  $\alpha_s$  and confirm that, as conjectured in Ref. [13], the leading double-logarithmic contributions to the flavour-singlet physical evolution kernels  $K_{\phi 2}$  and  $K_{2\phi}$  vanish at all orders. The key relation have been written down in  $N$ -space in Eqs. (20) and (29), but can be readily inverted back to  $x$ -space using the second part of Eq. (3), in that latter case using the series form (22).

The above results for the LL perturbative functions entering DIS (with space-like  $q^2 \equiv -Q^2$ ) can be carried over directly the time-like domain of semi-inclusive electron-positron annihilation or  $Z$  and Higgs-boson decay, which is related to the former case by a suitably defined (but for the present LL contribution essentially trivial) analytic continuation, see, e.g., Refs. [4, 25].

There is scope for improving upon the rigour of the present derivation of the crucial relations (13) and (14) in the future. One may expect such as improvement, and other interesting results, from the application of alternative approaches to deep-inelastic scattering, such as soft-collinear effective theory (SCET) [26] or the recent path-integral formulation for (sub-)leading threshold contributions [27]. Within the present framework an extension to at least the next-to-leading logarithms definitely appears feasible, and we plan to report on this issue in a later publication.

## Acknowledgments

I am grateful to S. Moch for pointing out Ref. [22] to me and for critically reading the manuscript. It is also a pleasure to thank J. Vermaseren for helpful comments concerning the effective high-order evaluation of nested expansions in FORM. This research has been supported by the UK Science & Technology Facilities Council (STFC) under grant numbers PP/E007414/1 and ST/G00062X/1.

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